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Symplectic quantization of self-dual master Lagrangian

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Abstract

We consider the master Lagrangian of Deser and Jackiw, interpolating between the self-dual and the Maxwell–Chern–Simons Lagrangian, and quantize it following the symplectic approach as well as the traditional Dirac scheme. We demonstrate the equivalence of these procedures in the subspace of the second-class constraints. We then proceed to embed this mixed first- and second-class system into an extended first-class system within the framework of both approaches, and construct the corresponding generator for this extended gauge symmetry in both formulations.

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1. Introduction

The traditional Dirac quantization method (DQM) [1] has been widely used in order to quantize Hamiltonian systems involving first- and second-class constraints. The resulting Dirac brackets defined on the subspace of the constraints may however be field dependent and nonlocal, and could thus pose serious ordering problems for the quantization of the theory. On the other hand, the Becchi–Rouet–Stora–Tyutin (BRST) [2, 3] procedure of first turning the second-class constraints into first-class ones along the lines originally established by Batalin, Fradkin and Vilkovisky [4, 5], and then reformulated in a more tractable and elegant version by Batalin, Fradkin and Tyutin (BFT) [6], does not suffer from these difficulties, as it relies on a simple Poisson bracket structure. As a result, the embedding of second-class systems into first-class ones (gauge theories) has received much attention in the past years, and the DQM improved in this way has been applied to a number of models [7–17] in order to obtain the corresponding Wess–Zumino (WZ) actions [18, 19].

On the other hand, the traditional Dirac approach [1] has been criticized for introducing ‘superfluous’ (primary) constraints. As a result, an alternative approach based on the symplectic structure of phase space has been proposed in [20]. The advantage of such

an approach in the case of first-order Lagrangians such as Chern–Simons theories has in particular been emphasized by Faddeev and Jackiw [20]. This symplectic scheme has been worked out in considerable detail in a series of papers [21], and has been applied to a number of models [21, 22]. It has further been extended recently to implement the improved DQM embedding program in the context of the symplectic formalism [23–25].

In this paper we illustrate, for the case of the master Lagrangian of Deser and Jackiw (DJ) [26], how the embedding of Hamiltonian systems with first- and second-class constraints into an extended gauge theory is realized in the context of the improved DQM, on one hand, and the improved symplectic approach, on the other.

The paper is organized as follows. In section 2, we briefly discuss the self-dual master model within the framework of the standard and the improved DQMs, where some of the degrees of freedom have been gauged. In section 3, we apply the gauge non-invariant symplectic formalism [20, 21] to this model, and in section 4 we then show how the improved DQM program for this master Lagrangian can be realized in the framework of the symplectic formalism by making a suitable ansatz for the Wess–Zumino term in the first-order formulation, respecting Lorentz invariance. Our conclusion is given in section 5. We leave it for appendix A to show how the improved DQM program can be used to turn all constraints of the DJ-master Lagrangian into first-class constraints. In appendix B we present the BFT construction of the involutive Hamiltonian of section 2 as an alternative to the procedure followed in that section. Finally, in appendix C we demonstrate the one-to-one correspondence between the symplectic approach and the Hamiltonian Dirac approach in the two cases.

2. Dirac quantization method

2.1. Standard Dirac quantization method

In this section, we consider the massive self-dual model Lagrangian [26]

$$\mathcal{L}_0 = \frac{m}{2} f_\mu f^\mu - \frac{1}{2} \epsilon_{\mu\nu\lambda} f^\mu \partial^\nu A^\lambda - \frac{1}{2} \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu f^\lambda + \frac{1}{2} \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda. \quad (2.1)$$

The canonical momenta conjugate to the fields f^μ and A^μ are given by

$$\begin{aligned} \pi_0^f &= 0 & \pi_i^f &= -\frac{1}{2} \epsilon_{ij} A^j \\ \pi_0^A &= 0 & \pi_i^A &= -\frac{1}{2} \epsilon_{ij} f^j + \frac{1}{2} \epsilon_{ij} A^j \end{aligned} \quad (2.2)$$

with the Poisson algebra $\{f^\mu(x), \pi_\nu^f(y)\} = \{A^\mu(x), \pi_\nu^A(y)\} = \delta_\nu^\mu \delta(x-y)$. The canonical Hamiltonian then reads

$$H_c = \int d^2x \left[-\frac{m}{2} f_\mu f^\mu + \epsilon_{ij} f^0 \partial^i A^j + \epsilon_{ij} A^0 (\partial^i f^j - \partial^j A^i) \right]. \quad (2.3)$$

The primary constraints following from the definition of the canonical momenta, are

$$\begin{aligned} \phi_0^f &\equiv \pi_0^f \approx 0 & \phi_i^f &\equiv \pi_i^f + \frac{1}{2} \epsilon_{ij} A^j \approx 0 \\ \phi_0^A &\equiv \pi_0^A \approx 0 & \phi_i^A &\equiv \pi_i^A + \frac{1}{2} \epsilon_{ij} f^j - \frac{1}{2} \epsilon_{ij} A^j \approx 0 \end{aligned} \quad (2.4)$$

with the corresponding primary Hamiltonian H_p ,

$$H_p = H_c + \int d^2x \sum_{\mu=0}^2 \left(v_f^\mu \phi_\mu^f + v_A^\mu \phi_\mu^A \right). \quad (2.5)$$

Persistence in time of the primary constraints leads to the secondary constraints

$$\varphi^f \equiv m f^0 - \epsilon_{ij} \partial^i A^j \approx 0 \quad \varphi^A \equiv -\epsilon_{ij} (\partial^i f^j - \partial^j A^i) \approx 0. \quad (2.6)$$

The constraints ϕ_i^f and ϕ_i^A fix the corresponding Lagrange multipliers to be $v_f^i = \partial^i f^0 + m\epsilon^{ij} f_j$ and $v_A^i = \partial^i A^0 + m\epsilon^{ij} f_j$, respectively. The Lagrange multiplier v_f^0 is determined by the time evolution of the constraint φ^f with the primary Hamiltonian to be $v_f^0 = \partial_i f^i$, while the multiplier v_A^0 remains undetermined. The fact of having an undetermined Lagrange multiplier v_A^0 reflects the existence of a gauge symmetry related to the fields A^μ . Indeed, with the redefinition [27] of the constraint $\varphi^A \rightarrow \omega^A = \varphi^A + \partial^i \phi_i^A$, we see that ω^A is first class and is the generator of the gauge transformation, $A^i \rightarrow A^i + \partial^i \lambda$.

We could now follow the BFT procedure in order to turn all constraints into first-class ones. This is left for appendix A. Here we are primarily interested in establishing the connection between the BFT embedding and the symplectic embedding procedures [25]. As it turns out (see sections 3 and 4), this connection is given in the subspace where the constraints $\phi_i^f = 0$ and $\phi_i^A = 0$ are implemented strongly. In this subspace, we are then left with two first-class constraints, $\phi_0^A \approx 0$, $\varphi^A \approx 0$, and two second-class ones, $\phi_0^f \approx 0$, $\varphi^f \approx 0$. We construct the corresponding Dirac brackets in terms of the inverse of the matrix Δ defined in terms of the Poisson brackets of $\{\phi_i^f, \phi_j^A\}$:

$$\Delta(x, y) = \begin{pmatrix} 0 & \epsilon \\ \epsilon & -\epsilon \end{pmatrix} \delta^2(x - y) \quad (2.7)$$

where

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2.8)$$

For the corresponding nonvanishing Dirac brackets, computed in the standard way, we have

$$\begin{aligned} \{A^0, \pi_0^A\}_D &= \delta^2(x - y) & \{f^0, \pi_0^f\}_D &= \delta^2(x - y) \\ \{f^i, f^j\}_D &= -\epsilon^{ij} \delta^2(x - y) & \{f^i, A^j\}_D &= -\epsilon^{ij} \delta^2(x - y) \\ \{A^i, \pi_j^A\}_D &= \frac{1}{2} \delta_j^i \delta^2(x - y) & \{f^i, \pi_j^f\}_D &= \frac{1}{2} \delta_j^i \delta^2(x - y) \\ \{f^i, \pi_j^A\}_D &= \frac{1}{2} \delta_j^i \delta^2(x - y) \end{aligned} \quad (2.9)$$

where we have used the convention $\epsilon_{12} = \epsilon^{12} = 1$ and $\epsilon_{ik} \epsilon^{kj} = -\delta_i^j$.

2.2. Improved Dirac quantization method

Following the improved DQM [7–16], we now proceed to embed the model into a gauge theory with respect to the above Dirac brackets, by extending phase space to include a pair of (canonically conjugate) auxiliary fields Φ^i , satisfying the Poisson brackets

$$\{\Phi^i(x), \Phi^j(y)\} = \epsilon^{ij} \delta^2(x - y). \quad (2.10)$$

Denote the second-class constraints (ϕ_0^f, φ^f) by $\{\Omega_i^f\}$, $i = 1, 2$. The first-class constraints $\tilde{\Omega}_i^f$ are now constructed as a power series in the auxiliary fields as follows,

$$\tilde{\Omega}_i^f = \Omega_i^f + \sum_{n=1}^{\infty} \Omega_i^{(n)} \quad (2.11)$$

where $\Omega_i^{(n)}$, $n = 1, \dots, \infty$ are homogeneous polynomials in the auxiliary fields Φ^i of degree n , to be determined by the requirement that the constraints $\tilde{\Omega}_i^f$ be strongly involutive:

$$\{\tilde{\Omega}_i^f(x), \tilde{\Omega}_j^f(y)\}_D = 0. \quad (2.12)$$

Making the ansatz,

$$\Omega_i^{(1)}(x) = \int d^2y X_{ij}(x, y) \Phi^j(y) \quad (2.13)$$

and substituting this ansatz into (2.11), the requirement (2.12) leads to the simple solution $X_{ij}(x, y) = \sqrt{m} \delta_{ij} \delta^2(x - y)$. There are no higher order contributions to (2.11). We thus obtain for the first-class constraints

$$\tilde{\phi}_0^f = \pi_0^f + m\theta \quad \tilde{\varphi}^f = m f^0 - \epsilon_{ij} \partial^i A^j + \pi_\theta \quad (2.14)$$

satisfying the first-class algebra $\{\tilde{\phi}_0^f, \tilde{\varphi}^f\} = 0$, where we have replaced (Φ^1, Φ^2) by $(\sqrt{m}\theta, \pi_\theta/\sqrt{m})$, for convenience.

Applying this procedure to the original field variables, we similarly obtain for the corresponding first-class fields

$$\tilde{f}^0 = f^0 + \frac{1}{m} \pi_\theta \quad \tilde{f}^i = f^i + \partial^i \theta \quad \tilde{\pi}_0^f = \pi_0^f + m\theta \quad (2.15)$$

satisfying $\{F(\tilde{f}^\mu, \tilde{\pi}_0^f), \tilde{\Omega}_i^f\} = 0$.

Since an arbitrary functional of the first-class fields is also first class, we can obtain the first-class Hamiltonian \tilde{H}_c by simply replacing the original fields by the respective tilde-fields [8, 28],

$$\begin{aligned} \tilde{H}_c = \int d^2x \left[-\frac{m}{2} \left(f^0 + \frac{1}{m} \pi_\theta \right)^2 + \frac{m}{2} (f^i + \partial^i \theta)^2 \right. \\ \left. + \epsilon_{ij} \left(f^0 + \frac{1}{m} \pi_\theta \right) \partial^i A^j + \epsilon_{ij} A^0 (\partial^i f^j - \partial^j A^i) \right] \end{aligned} \quad (2.16)$$

along with the remaining first-class constraints now written in the extended phase space as

$$\tilde{\phi}_0^A = \tilde{\pi}_0^A = \pi_0^A \quad \tilde{\varphi}^A = -\epsilon_{ij} \partial^i \tilde{f}^j + \epsilon_{ij} \partial^i \tilde{A}^j = -\epsilon_{ij} \partial^i f^j + \epsilon_{ij} \partial^i A^j. \quad (2.17)$$

Note that we have taken here $\tilde{A}^\mu = A^\mu$. Indeed, A^μ and π_0^A remain unchanged by the embedding procedure, which only involves the f^μ -fields. A^μ thus continues to transform as usual under gauge transformations, and is not first class. One may thus question our simple substitution procedure for arriving at the first-class Hamiltonian (2.16). In appendix B we show how this Hamiltonian is obtained following the usual BFT construction of the involutive Hamiltonian [6].

Now, let us streamline the notation by defining

$$(\tilde{\Omega}_\alpha^f) = (\tilde{\phi}_0^f, \tilde{\varphi}^f) \quad (\tilde{\Omega}_\alpha^A) = (\tilde{\phi}_0^A, \tilde{\varphi}^A). \quad (2.18)$$

With respect to the Dirac brackets defined previously, we then have the relations of strong involution,

$$\begin{aligned} \{\tilde{\Omega}_\alpha^f, \tilde{\Omega}_\beta^A\}_D = 0 \quad \{\tilde{\Omega}_\alpha^f, \tilde{H}_c\}_D = 0 \\ \{\tilde{\phi}_0^A, \tilde{H}_c\}_D = \tilde{\varphi}^A \quad \{\tilde{\varphi}^A, \tilde{H}_c\}_D = 0. \end{aligned} \quad (2.19)$$

Note that the first-class Hamiltonian (2.16) does not generate naturally the first-class Gauss law constraints from the time evolution of the primary constraints $\tilde{\phi}_0^f \approx 0$, $\tilde{\phi}_0^A \approx 0$. For this to be the case, we introduce an additional term proportional to the first-class constraints $\tilde{\phi}_0^f$ into the Hamiltonian density $\tilde{\mathcal{H}}_c$, leading us to consider the equivalent first-class Hamiltonian

$$\tilde{\mathcal{H}}'_c = \tilde{\mathcal{H}}_c + \frac{1}{m} \pi_\theta \tilde{\phi}_0^f. \quad (2.20)$$

We then obtain the Dirac brackets in the desired form:

$$\begin{aligned} \{\tilde{\varphi}_0^f(x), \tilde{H}'_c\}_D &= \tilde{\varphi}^f(x) & \{\tilde{\varphi}^f(x), \tilde{H}'_c\} &= 0 \\ \{\tilde{\varphi}_0^A(x), \tilde{H}'_c\}_D &= \tilde{\varphi}^A(x) & \{\tilde{\varphi}^A(x), \tilde{H}'_c\} &= 0. \end{aligned} \quad (2.21)$$

We streamline further the notation by collecting all the first-class constraints into a single vector:

$$\tilde{\Omega}_A = (\tilde{\Omega}_\alpha^f, \tilde{\Omega}_\alpha^A). \quad (2.22)$$

Note here that the subscript A is the index running 1 to 4, while the superscript A in $\tilde{\Omega}_\alpha^A$ denotes that these constraints are related to the field A^μ in the model.

We now seek the equivalent Lagrangian corresponding to the first-class Hamiltonian $\tilde{\mathcal{H}}'_c$ in (2.20). To this end, we consider the partition function in the phase space as given by the Faddeev–Senjanovic prescription [29],

$$Z = N \int \mathcal{D}f^\mu \mathcal{D}A^\mu \mathcal{D}\theta \mathcal{D}\pi_\theta \mathcal{D}\pi_0^f \mathcal{D}\pi_0^A \prod_{A,B} \delta(\tilde{\Omega}_A) \delta(\Gamma_B) \det|\{\tilde{\Omega}_A, \Gamma_B\}| e^{i \int d^3x \mathcal{L}} \quad (2.23)$$

$$\mathcal{L} = \pi_0^f \dot{f}^0 - \frac{1}{2} \epsilon_{ij} A^j \dot{f}^i + \pi_0^A \dot{A}^0 + \left(-\frac{1}{2} \epsilon_{ij} f^j + \frac{1}{2} \epsilon_{ij} A^j\right) \dot{A}^i + \pi_\theta \dot{\theta} - \tilde{\mathcal{H}}'_c$$

where the gauge fixing conditions Γ_B are chosen so that the determinant occurring in the functional measure is nonvanishing.

Exponentiating the delta function $\delta(\tilde{\varphi}^f)$ as $\delta(\tilde{\varphi}^f) = \int \mathcal{D}\xi e^{i \int d^3x \xi \tilde{\varphi}^f}$, making a transformation $f^0 \rightarrow f^0 + \xi$ and performing the integration over π_0^f , π_θ , π_0^A and ξ , the partition function (2.23) reduces to

$$Z = N \int \mathcal{D}f^\mu \mathcal{D}A^\mu \mathcal{D}\theta \prod_{A,B} \delta(\Gamma_B) \det|\{\tilde{\Omega}_A, \Gamma_B\}| e^{i \int d^3x \mathcal{L}} \quad (2.24)$$

where

$$\begin{aligned} \mathcal{L} = \frac{m}{2} (f_\mu + \partial_\mu \theta)(f^\mu + \partial^\mu \theta) - \frac{1}{2} \epsilon_{\mu\nu\lambda} (f^\mu + \partial^\mu \theta) \partial^\nu A^\lambda \\ - \frac{1}{2} \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu (f^\lambda + \partial^\lambda \theta) + \frac{1}{2} \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda \end{aligned} \quad (2.25)$$

is the manifestly gauge invariant Stückelberg Lagrangian with the Stückelberg scalar θ .

Next, we construct the generator G of gauge transformations, following Dirac's conjecture [1], for the embedded self-dual master model in the standard way,

$$G = \int d^3x \sum_\alpha [\epsilon_\alpha^f \tilde{\Omega}_\alpha^f + \epsilon_\alpha^A \tilde{\Omega}_\alpha^A] \quad (2.26)$$

where $\epsilon_\alpha^f, \epsilon_\alpha^A$ are, in general, functions of phase space variables and $\tilde{\Omega}_\alpha^f, \tilde{\Omega}_\alpha^A$ are the first-class constraints in equation (2.18). The infinitesimal gauge transformation for a function F of phase space variables is then given by the relation of $\delta F = \{F, G\}_D$, and leads to

$$\delta f^0 = \epsilon_1^f \quad \delta f^i = -\partial^i \epsilon_2^f \quad \delta A^0 = \epsilon_1^A \quad \delta A^i = -\partial^i \epsilon_2^A \quad \delta \theta = \epsilon_2^f. \quad (2.27)$$

The above gauge transformation involving four gauge parameters is a symmetry of the Hamiltonian, but not of the Lagrangian. The generator G of the most general local symmetry transformation of a Lagrangian must satisfy the master equation [30]

$$\frac{\partial G}{\partial t} + \{G, H_T\} = 0 \quad (2.28)$$

which, together with (2.26), implies the following well-known restrictions on the gauge parameters and on the Lagrange multipliers in the primary Hamiltonian:

$$\delta v^\beta = \frac{d\epsilon^\beta}{dt} - \epsilon^A \left(V_A^\beta + v^\alpha C_{\alpha A}^\beta \right) \quad 0 = \frac{d\epsilon^b}{dt} - \epsilon^A \left(V_A^b + v^\alpha C_{\alpha A}^b \right). \quad (2.29)$$

Here the superscripts α, β (a, b) denote the primary (secondary) constraints, and V_B^A, C_{BC}^A are the structure functions of the constrained Hamiltonian dynamics defined by $\{H_c, \tilde{\Omega}_A\}_D = V_A^B \tilde{\Omega}_B, \{\tilde{\Omega}_A, \tilde{\Omega}_B\}_D = C_{AB}^C \tilde{\Omega}_C$, respectively. From (2.29) we obtain $\epsilon_1^f = -d\epsilon_2^f/dt$ and $\epsilon_1^A = -d\epsilon_2^A/dt$. Thus the gauge transformations (2.27) reduce to

$$\delta f^\mu = -\partial^\mu \epsilon_2^f \quad \delta A^\mu = -\partial^\mu \epsilon_2^A \quad \delta \theta = \epsilon_2^f \quad (2.30)$$

which evidently leaves the Stückelberg Lagrangian (2.25) invariant.

3. Constraint structure of master Lagrangian in symplectic approach

In this and the following sections, we show that the results obtained in section 2 are in full agreement with those obtained in the symplectic approach. We begin by considering the symplectic analogue of the conventional Dirac approach.

The master Lagrangian (2.1) is of the form

$$L = \int d^2x a(x)_\alpha \dot{\xi}_\alpha(x) - V[\xi] \quad (3.1)$$

where

$$(\xi_\alpha) = (f^1, f^2, A^1, A^2, f^0, A^0) \quad (3.2)$$

$$(a_\alpha) = \left(-\frac{1}{2}A^2, \frac{1}{2}A^1, -\frac{1}{2}(f^2 - A^2), \frac{1}{2}(f^1 - A^1), 0, 0\right) \quad (3.3)$$

and

$$V = \int d^2x \left[-\frac{m}{2} f_\mu f^\mu + f^0 \epsilon_{ij} \partial^i A^j + A^0 (\epsilon_{ij} \partial^i f^j - \epsilon_{ij} \partial^i A^j) \right]. \quad (3.4)$$

The Euler–Lagrange equations then read

$$\int d^2y F_{\alpha\beta}^{(0)}(x, y) \dot{\xi}_\beta(y) = K_\alpha^{(0)}(x) \quad (3.5)$$

where

$$(K_\alpha^{(0)}) = \frac{\delta V}{\delta \xi_\alpha(x)} = \begin{pmatrix} \partial^2 A^0 + m f^1 \\ -\partial^1 A^0 + m f^2 \\ -\partial_2(f^0 - A^0) \\ \partial_1(f^0 - A^0) \\ \epsilon_{ij} \partial^i A^j - m f^0 \\ \epsilon_{ij} \partial^i (f^j - A^j) \end{pmatrix} \quad (3.6)$$

and $F_{\alpha\beta}^{(0)}$ is the (pre-)symplectic form [21]

$$F_{\alpha\beta}^{(0)}(x, y) = \frac{\partial a_\beta(y)}{\partial \xi_\alpha(x)} - \frac{\partial a_\alpha(x)}{\partial \xi_\beta(y)}. \quad (3.7)$$

Explicitly,

$$F^{(0)}(x, y) = \begin{pmatrix} 0 & \epsilon & 0 \\ \epsilon & -\epsilon & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta^2(x - y) \quad (3.8)$$

where ϵ is the matrix (2.8), and 0 is the 2×2 matrix. It is evident that since $\det F^{(0)} = 0$, the matrix $F^{(0)}$ is not invertible. In fact, the rank of this matrix is 4, so that there exists two-fold

infinity of zero-generation (left) zero modes $u^{(0)}(\sigma; z)$, labelled by discrete indices $\sigma = 1, 2$ and the continuum label z , with components

$$u_x^{(0)T}(1; z) = (0, 0, 0, 0, -1, 0)\delta^2(x - z) \quad u_x^{(0)T}(2; z) = (0, 0, 0, 0, 0, -1)\delta^2(x - z) \quad (3.9)$$

where the superscript T stands for transpose. Correspondingly, we have a two-fold infinity of ‘zero generation’ constraints

$$\varphi_\sigma(z) = \int d^2x \sum_\alpha u_{\alpha,x}^{(0)}(\sigma, z) \frac{\delta V}{\delta \xi_\alpha(x)} = 0. \quad (3.10)$$

Explicitly,

$$\begin{aligned} \varphi_1(z) &= -\frac{\delta V}{\delta f^0(z)} = m f^0(z) - \epsilon_{ij} \partial^i A^j(z) \\ \varphi_2(z) &= -\frac{\delta V}{\delta A^0(z)} = -\epsilon_{ij} \partial^i (f^j(z) - A^j(z)). \end{aligned} \quad (3.11)$$

Comparing with (2.6) we see that $\varphi_1 = \varphi^f$ and $\varphi_2 = \varphi^A$. We must require these constraints to be conserved in time,

$$\partial_0 \varphi_\sigma(z) = 0 \quad (3.12)$$

or

$$\int d^2x \sum_\alpha \frac{\partial \varphi_\sigma(z)}{\partial \xi_\alpha(x)} \dot{\xi}_\alpha(x) = 0. \quad (3.13)$$

These equations of motion are obtained as one of the Euler–Lagrange equations of the extended Lagrangian³

$$L' = L - \int d^2x \sum_\sigma \varphi_\sigma(z) \dot{\eta}_\sigma(z). \quad (3.14)$$

The field A^0 only occurs in the potential V in the form $A^0(z)\varphi_2(z)$. Hence it can be absorbed into a new dynamical variable via the shift $\dot{\eta}_2 - A^0 \rightarrow \dot{\eta}_2$. Our new set of ‘first-generation’ dynamical variables are then

$$(\xi_{\alpha_1}^{(1)}) = (f^1, f^2, A^1, A^2, f^0, \eta_1, \eta_2) \quad (3.15)$$

and the ‘first-generation’ Lagrangian takes the form

$$L^{(1)} = \int d^2x \sum_{\alpha_1=1}^7 a_{\alpha_1}^{(1)} \dot{\xi}_{\alpha_1}^{(1)} - V^{(1)}[\xi] \quad (3.16)$$

where

$$(a_{\alpha_1}^{(1)}(x)) = \left(-\frac{1}{2}A^2, \frac{1}{2}A^1, -\frac{1}{2}(f^2 - A^2), \frac{1}{2}(f^1 - A^1), 0, -\varphi_1(x), -\varphi_2(x)\right) \quad (3.17)$$

and

$$V^{(1)} = \int d^2x \left[-\frac{m}{2} f_\mu f^\mu + f^0 \epsilon_{ij} \partial^i A^j\right]. \quad (3.18)$$

The equations of motion now take the form

$$\int d^2y F_{\alpha_1, \beta_1}^{(1)}(x, y) \dot{\xi}_{\beta_1}^{(1)}(y) = \frac{\delta V^{(1)}}{\delta \xi_{\alpha_1}^{(1)}(x)} \quad (3.19)$$

³ The minus sign is chosen for later convenience, when comparing with the Dirac quantization procedure.

where the ‘first-generation’ symplectic form $F_{\alpha_1, \beta_1}^{(1)}$ is given by

$$F_{\alpha_1 \beta_1}^{(1)}(x, y) = \frac{\partial a_{\beta_1}^{(1)}(y)}{\partial \xi_{\alpha_1}^{(1)}(x)} - \frac{\partial a_{\alpha_1}^{(1)}(x)}{\partial \xi_{\beta_1}^{(1)}(y)} \quad (3.20)$$

or explicitly

$$F^{(1)}(x, y) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & -\partial_2 \\ 0 & 0 & -1 & 0 & 0 & 0 & \partial_1 \\ 0 & 1 & 0 & -1 & 0 & -\partial_2 & \partial_2 \\ -1 & 0 & 1 & 0 & 0 & \partial_1 & -\partial_1 \\ 0 & 0 & 0 & 0 & 0 & -m & 0 \\ 0 & 0 & \partial_2 & -\partial_1 & m & 0 & 0 \\ \partial_2 & -\partial_1 & -\partial_2 & \partial_1 & 0 & 0 & 0 \end{pmatrix} \delta^2(x - y). \quad (3.21)$$

$F^{(1)}$ exhibits one zero mode,

$$u_x^{(1)T}(z) = (0, 0, \partial_1, \partial_2, 0, 0, 1) \delta^2(x - z). \quad (3.22)$$

Noting that

$$(K_\alpha^{(1)}) = \frac{\delta V^{(1)}}{\delta \xi_\alpha} = \begin{pmatrix} mf^1 \\ mf^2 \\ -\partial_2 f^0 \\ \partial_1 f^0 \\ \epsilon_{ij} \partial^i A^j - mf^0 \\ 0 \\ 0 \end{pmatrix} \quad (3.23)$$

we find that the new constraint vanishes identically:

$$\int d^2z u_{\alpha_1, x}^{(1)}(z) \frac{\delta V^{(1)}}{\delta \xi_{\alpha_1}^{(1)}(z)} = (\partial_1 \partial_2 f^0 - \partial_1 \partial_2 f^0) \equiv 0. \quad (3.24)$$

Hence the algorithm ends at this point. We now write $F^{(1)}$ in the form

$$F^{(1)}(x, y) = \begin{pmatrix} f & M \\ -M^T & 0 \end{pmatrix} \delta^2(x - y) \quad (3.25)$$

where

$$f(x, y) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -\partial_2 \\ -1 & 0 & 1 & 0 & 0 & \partial_1 \\ 0 & 0 & 0 & 0 & 0 & -m \\ 0 & 0 & \partial_2 & -\partial_1 & m & 0 \end{pmatrix} \delta^2(x - y) \quad (3.26)$$

and M is the 1×6 matrix

$$M_{\alpha_1}(x, y) = \begin{pmatrix} -\frac{\partial \varphi_2(y)}{\partial \xi_{\alpha_1}^{(1)}(x)} \end{pmatrix} = \begin{pmatrix} -\partial_2 \\ \partial_1 \\ \partial_2 \\ -\partial_1 \\ 0 \\ 0 \end{pmatrix} \delta^2(x - y). \quad (3.27)$$

We next observe that $\det f \neq 0$, so that the inverse of f exists. It is readily computed to be

$$f^{-1}(x, y) = \begin{pmatrix} 0 & -1 & 0 & -1 & \frac{1}{m}\partial_1 & 0 \\ 1 & 0 & 1 & 0 & \frac{1}{m}\partial_2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{m}\partial_1 & -\frac{1}{m}\partial_2 & 0 & 0 & 0 & \frac{1}{m} \\ 0 & 0 & 0 & 0 & -\frac{1}{m} & 0 \end{pmatrix} \delta^2(x - y). \quad (3.28)$$

The zero mode (3.22) is of the general form [21]

$$u_{\alpha_1, x}^{(1)}(z) = \left(\sum_{B=1}^6 \int d^2 y f_{AB}^{-1}(x, y) \frac{\partial \varphi_2(z)}{\partial \xi_B^{(1)}(y)}, 1 \right) \quad A, B = 1, \dots, 6 \quad (3.29)$$

where we label the subspace on which f^{-1} is defined by the indices A, B, C, \dots . As we shall see, in the algorithm of Dirac these label the complete set of second-class constraints. The zero mode (3.22) is the generator of gauge transformations in the sense [21] that

$$\delta \xi_{\alpha_1}^{(1)}(x) = \int d^2 z u_{\alpha_1, x}^{(1)}(z) \epsilon(z). \quad (3.30)$$

With the aid of (3.29) we can readily rewrite this in terms of symplectic brackets. Let F and G be functions of the *dynamical* field variables ξ_A . We define generalized symplectic structures by

$$\{F, G\}^* = \int d^2 z \int d^2 z' \frac{\partial F}{\partial \xi_A(z)} f_{AB}^{-1}(z, z') \frac{\delta G}{\delta \xi_B(z')}. \quad (3.31)$$

In particular,

$$\{\xi_A(x), \xi_B(y)\}^* = f_{AB}^{-1}(x, y) \quad (3.32)$$

or explicitly

$$\{f^i(x), f^j(y)\}^* = -\epsilon^{ij} \delta^2(x - y) \quad \{f^i(x), A^j(y)\}^* = -\epsilon^{ij} \delta^2(x - y)$$

in agreement with the Dirac brackets in equation (2.9).

In terms of the symplectic structure (3.31) we may write (3.30) in a form which will be convenient for later comparison:

$$\begin{aligned} (\delta \xi_{\alpha_1}^{(1)}(x)) &= \left(\int d^2 y \int d^2 z f_{AB}^{-1}(x, y) \frac{\partial \varphi_2(z)}{\partial \xi_B^{(1)}(y)} \epsilon(z), \epsilon(x) \right) \\ &= \left(\int d^2 z \left\{ \xi_A^{(1)}(x), \varphi_2(z) \right\}^* \epsilon(z), \epsilon(x) \right). \end{aligned} \quad (3.33)$$

Explicitly,

$$\begin{aligned} \delta A^1 &= -\partial^1 \epsilon & \delta A^2 &= -\partial^2 \epsilon \\ \delta f^0 &= 0 & \delta f^1 &= 0 & \delta f^2 &= 0 \\ \delta \eta_1 &= 0 & \delta \eta_2 &= \epsilon. \end{aligned} \quad (3.34)$$

Recalling the redefinition $\hat{\eta}_2 - A^0 \rightarrow \hat{\eta}_2$, we see that $\delta \eta_2 = \epsilon$ implies $\delta A^0 = -\partial^0 \epsilon$, in agreement with our expectations.

In appendix C we show that our results are in one-to-one correspondence with the conventional Dirac algorithm.

4. Symplectic embedding of master Lagrangian

It is interesting to examine the relation between the BFT embedding procedure (improved Dirac approach) discussed in section 2 and a corresponding embedding in the symplectic formulation. The procedure of section 2 has led to a Stückelberg Lagrangian, where the field f^μ has been gauged by the introduction of a Stückelberg scalar θ . This suggests the introduction of an additive ‘Wess–Zumino’ term to the Lagrangian density (3.1) of the Lorentz covariant form

$$\mathcal{L}_{\text{WZ}} = \alpha f^\mu \partial_\mu \theta + \frac{\beta}{2} \partial_\mu \theta \partial^\mu \theta.$$

Following a standard recipe [25, 26] for constructing the corresponding first-order Lagrangian, one readily checks that the corresponding equivalent symplectic Lagrangian now reads as in (3.1), with

$$(\xi_\alpha) = (f^1, f^2, A^1, A^2, \theta, \pi_\theta, f^0, A^0) \quad (4.1)$$

$$(a_\alpha) = \left(-\frac{1}{2}A^2, \frac{1}{2}A^1, -\frac{1}{2}(f^2 - A^2), \frac{1}{2}(f^1 - A^1), \pi_\theta, 0, 0, 0\right) \quad (4.2)$$

and

$$V[A, f, \theta, \pi_\theta] = \int d^2x \left[-\frac{m}{2} f_\mu f^\mu + f^0 \epsilon_{ij} \partial^i A^j + A^0 (\epsilon_{ij} \partial^i f^j - \epsilon_{ij} \partial^i A^j) + \frac{1}{2\beta} (\pi_\theta - \alpha f^0)^2 - \alpha f^i \partial_i \theta - \frac{\beta}{2} \partial^i \theta \partial_i \theta \right]. \quad (4.3)$$

The Euler–Lagrange equations then read as in (3.5), with $K^{(0)}$ replaced by

$$(K_\alpha^{(0)}) = \frac{\delta V}{\delta \xi_\alpha(x)} = \begin{pmatrix} \partial^2 A^0 + m f^1 + \alpha \partial^1 \theta \\ -\partial^1 A^0 + m f^2 + \alpha \partial^2 \theta \\ -\partial_2 (f^0 - A^0) \\ \partial_1 (f^0 - A^0) \\ \alpha \partial_i f^i + \beta \partial_i \partial^i \theta \\ \frac{1}{\beta} (\pi - \alpha f^0) \\ \epsilon_{ij} \partial^i A^j - \frac{\alpha}{\beta} (\pi - \alpha f^0) - m f^0 \\ \epsilon_{ij} \partial^i (f^j - A^j) \end{pmatrix} \quad (4.4)$$

and $F_{\alpha\beta}^{(0)}$ the (pre-)symplectic form now replaced by

$$F^{(0)}(x, y) = \begin{pmatrix} 0 & \epsilon & 0 & 0 \\ \epsilon & -\epsilon & 0 & 0 \\ 0 & 0 & -\epsilon & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta^2(x - y). \quad (4.5)$$

As is again evident, since $\det F^{(0)} = 0$, the matrix $F^{(0)}$ is not invertible. In fact, the rank of this matrix is 6, so that there exists two-fold infinity of zero-generation (left) zero modes $u^{(0)}(\sigma, z)$, labelled by discrete indices $\sigma = 1, 2$ and the continuum label z , with components

$$\begin{aligned} u_x^{(0)T}(1; z) &= (0, 0, 0, 0, 0, 0, -1, 0) \delta^2(x - z) \\ u_x^{(0)T}(2; z) &= (0, 0, 0, 0, 0, 0, 0, -1) \delta^2(x - z) \end{aligned} \quad (4.6)$$

implying the constraints

$$\begin{aligned} \varphi_1(z) &= -\frac{\delta V}{\delta f^0(z)} = m f^0(z) - \epsilon_{ij} \partial^i A^j(z) + \frac{\alpha}{\beta} (\pi_\theta(z) - \alpha f^0(z)) \\ \varphi_2(z) &= -\frac{\delta V}{\delta A^0(z)} = -\epsilon_{ij} \partial^i (f^j(z) - A^j(z)). \end{aligned} \quad (4.7)$$

We proceed as in section 3, being led to a first-level Lagrangian

$$L^{(1)} = \int d^2x \sum_{\alpha_1=1}^9 a_{\alpha_1}^{(1)} \dot{\xi}_{\alpha_1}^{(1)} - V^{(1)}[\xi] \quad (4.8)$$

with a new set of ‘first-generation’ dynamical variables in which A^0 has again been absorbed into a redefinition of η_2 ,

$$(\xi_{\alpha_1}^{(1)}) = (f^1, f^2, A^1, A^2, \theta, \pi_\theta, f^0, \eta_1, \eta_2) \quad (4.9)$$

and

$$(a_{\alpha_1}^{(1)}(x)) = \left(-\frac{1}{2}A^2, \frac{1}{2}A^1, -\frac{1}{2}(f^2 - A^2), \frac{1}{2}(f^1 - A^1), \pi_\theta, 0, 0, -\varphi_1(x), -\varphi_2(x)\right) \quad (4.10)$$

and

$$V^{(1)}[A, f, \theta, \pi_\theta] = \int d^2x \left[-\frac{m}{2} f_\mu f^\mu + f^0 \epsilon_{ij} \partial^i A^j + \frac{1}{2\beta} (\pi_\theta - \alpha f^0)^2 - \alpha f^i \partial_i \theta - \frac{\beta}{2} \partial^i \theta \partial_i \theta \right]. \quad (4.11)$$

The equations of motion now take the form (3.19), where the ‘first-generation’ symplectic form $F_{\alpha_1, \beta_1}^{(1)}$ is now given by (3.20), with

$$F^{(1)}(x, y) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\partial_2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \partial_1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & -\partial_2 & \partial_2 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & \partial_1 & -\partial_1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\frac{\alpha}{\beta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\kappa & 0 \\ 0 & 0 & \partial_2 & -\partial_1 & 0 & \frac{\alpha}{\beta} & \kappa & 0 & 0 \\ \partial_2 & -\partial_1 & -\partial_2 & \partial_1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \delta^2(x - y) \quad (4.12)$$

where $\kappa = m - \alpha^2/\beta$, and $K^{(1)}$ is given by

$$(K_\alpha^{(1)}) = \frac{\delta V^{(1)}}{\delta \xi_\alpha} = \begin{pmatrix} m f^1 + \alpha \partial^1 \theta \\ m f^2 + \alpha \partial^2 \theta \\ -\partial_2 f^0 \\ \partial_1 f^0 \\ \alpha \partial_i f^i + \beta \partial_i \partial^i \theta \\ \frac{1}{\beta} (\pi_\theta - \alpha f^0) \\ \epsilon_{ij} \partial^i A^j - \frac{\alpha}{\beta} (\pi_\theta - \alpha f^0) - m f^0 \\ 0 \\ 0 \end{pmatrix}. \quad (4.13)$$

$F^{(1)}$ exhibits one zero mode,

$$u_x^{(1)T}(1; z) = (0, 0, \partial_1, \partial_2, 0, 0, 0, 0, 1) \delta^2(x - z) \quad (4.14)$$

implying however an identically vanishing constraint. For $\frac{\alpha^2}{\beta} = m$ there are two further zero modes:

$$u_x^{(1)T}(2; z) = (0, 0, 0, 0, 0, 0, 1, 0, 0) \delta^2(x - z) \quad (4.15)$$

$$u_x^{(1)T}(3; z) = \left(\partial_1, \partial_2, 0, 0, \frac{\alpha}{\beta}, 0, 0, 1, 0 \right) \delta^2(x - z). \quad (4.16)$$

The zero modes $u_x^{(1)T}(2; z)$ and $u_x^{(2)T}(3; z)$ both imply the constraint $\varphi_1 = 0$ in (4.7), now evaluated for $\kappa = m - \frac{\alpha^2}{\beta} = 0$:

$$\varphi_1 \equiv \epsilon_{ij} \partial^i A^j - \frac{\alpha}{\beta} \pi_\theta. \quad (4.17)$$

The corresponding potential (4.11) now takes the form

$$V^{(1)}[A, f, \theta, \pi_\theta] = \int d^2x \left[-\frac{m}{2} f_i f^i + \frac{1}{2\beta} \pi_\theta^2 - \alpha f^i \partial_i \theta - \frac{\beta}{2} \partial^i \theta \partial_i \theta - f^0 \varphi_1 \right]. \quad (4.18)$$

We may thus absorb the term $f^0 \varphi_1$ in the potential into a redefinition of the variable η_1 : $\dot{\eta}_1 - f^0 \rightarrow \dot{\eta}_1$. Correspondingly,

$$(\xi_{\alpha_1}^{(1)}) = (f^1, f^2, A^1, A^2, \theta, \pi_\theta, \eta_1, \eta_2) \quad (4.19)$$

and $K^{(1)}$ is given by

$$(K_{\alpha_1}^{(1)}) = \frac{\delta V^{(1)}}{\delta \xi_{\alpha_1}} = \begin{pmatrix} mf^1 + \alpha \partial^1 \theta \\ mf^2 + \alpha \partial^2 \theta \\ -\partial_2 f^0 \\ \partial_1 f^0 \\ \alpha \partial_i f^i + \beta \partial_i \partial^i \theta \\ \frac{1}{\beta} (\pi_\theta - \alpha f^0) \\ 0 \\ 0 \end{pmatrix} \quad (4.20)$$

$$F^{(1)}(x, y) = \begin{pmatrix} f & M \\ -M^T & 0 \end{pmatrix} \delta^2(x - y) \quad (4.21)$$

where

$$f(x, y) = \begin{pmatrix} 0 & \epsilon & 0 \\ \epsilon & -\epsilon & 0 \\ 0 & 0 & -\epsilon \end{pmatrix} \quad (4.22)$$

and M is the 2×6 matrix

$$M_{\alpha_1}(x, y) = \begin{pmatrix} 0 & -\partial_2 \\ 0 & \partial_1 \\ -\partial_2 & \partial_2 \\ \partial_1 & -\partial_1 \\ 0 & 0 \\ -\frac{\alpha}{\beta} & 0 \end{pmatrix} \delta^2(x - y). \quad (4.23)$$

In the form (4.21), $F^{(1)}$ still has two zero modes:

$$u_x^f(z) = \left(\partial_1, \partial_2, 0, 0, \frac{\alpha}{\beta}, 0, 1, 0 \right) \delta^2(x - z) \quad (4.24)$$

$$u_x^A(z) = (0, 0, \partial_1, \partial_2, 0, 0, 0, 1) \delta^2(x - z).$$

One readily checks that they imply identically vanishing constraints:

$$u^f(z) \cdot K^{(1)} \equiv 0 \quad u^A(z) \cdot K^{(1)} \equiv 0.$$

They therefore generate the following gauge transformation:

$$\delta \xi_{\alpha_1}(x) = \int d^2z \left[u_{\alpha_1, x}^f(z) \epsilon^f(z) + u_{\alpha_1, x}^A(z) \epsilon^A(z) \right] \quad (4.25)$$

or explicitly

$$\begin{aligned} \delta A^1 &= -\partial^1 \epsilon^A & \delta A^2 &= -\partial^2 \epsilon^A \\ \delta f^1 &= -\partial^1 \epsilon^f & \delta f^2 &= -\partial^2 \epsilon^f \\ \delta \theta &= \frac{\alpha}{\beta} \epsilon^f & \delta \pi_\theta &= 0 \\ \delta \eta_1 &= \epsilon^f & \delta \eta_2 &= \epsilon^A \end{aligned} \quad (4.26)$$

Recalling the relabelling $\dot{\eta}_1 - f^0 \rightarrow \dot{\eta}_1$ and $\dot{\eta}_2 - A^0 \rightarrow \dot{\eta}_2$, we see that the transformations for η_i imply

$$\delta f^0 = -\partial^0 \epsilon^f \quad \delta A^0 = -\partial^0 \epsilon^A$$

in accordance with our expectations: $\delta f^\mu = -\partial^\mu \epsilon^f$, $\delta A^\mu = -\partial^\mu \epsilon^A$ and $\delta \theta = \epsilon^f$ which are exactly the same as transformation (2.30) obtained from the improved DQM when we assign the coefficients for $\alpha = \beta = m$.

5. Conclusion

It has been the primary objective of this paper to illustrate in terms of a non-trivial model as described by the master Lagrangian of Deser and Jackiw [26], how the embedding of Hamiltonian systems with first- and second-class constraints into an extended gauge theory is realized in the context of the improved Dirac quantization method (DQM) on one hand, and the improved symplectic approach, on the other.

Alternatively, rather than proceeding iteratively as one does in the improved DQM approach, we have simplified the calculation in the symplectic case by making use of manifest Lorentz invariance in our ansatz for the Wess–Zumino term to be added to the master Lagrangian, and then reformulating the problem in terms of an equivalent first-order Lagrangian.

Just as in the case of the improved DQM procedure, the symplectic embedding procedure requires the introduction of an even number of additional fields, which, following the Faddeev–Jackiw prescription [20], can be chosen to be canonically conjugate pairs. This is in line with the fact that the number of second-class constraints is always even, and that the improved DQM embedding procedure requires that phase space be augmented by one degree of freedom for each secondary constraint. This fact has not been recognized in a recent paper on the subject [24], where in our notation, π_θ is effectively taken to be a function of A^i , π_i and θ .

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Appendix A. BFT-embedding into a fully first-class system

In this appendix, we demonstrate how the improved DQM can be used in order to turn the model defined by the master Lagrangian into a fully first-class system on the Hamiltonian level. In order to extract the true second-class constraints, we redefine the secondary constraint φ^A as follows:

$$\omega^A = \partial^i \pi_i^A - \frac{1}{2} \epsilon_{ij} \partial^i f^j + \frac{1}{2} \epsilon_{ij} \partial^i A^j. \quad (A.1)$$

The nonvanishing Poisson brackets are then given as

$$\begin{aligned}\{\phi_0^f(x), \varphi^f(y)\} &= -m\delta^2(x-y) \\ \{\phi_i^f(x), \phi_j^A(y)\} &= \epsilon_{ij}\delta^2(x-y) \\ \{\phi_i^A(x), \phi_j^A(y)\} &= -\epsilon_{ij}\delta^2(x-y)\end{aligned}\quad (\text{A.2})$$

which show that ϕ_0^A and ω^A are first class.

Redefining the constraints,

$$(\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6) = (\phi_0^f, \varphi^f, \phi_1^f, \phi_2^f, \phi_1^A, \phi_2^A) \quad (\text{A.3})$$

we obtain the second-class algebra

$$\Delta_{\alpha\beta} = \{\Omega_\alpha, \Omega_\beta\} = \begin{pmatrix} -m\epsilon & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & \epsilon & -\epsilon \end{pmatrix} \delta^2(x-y) \quad (\text{A.4})$$

where ϵ is the Levi-Civita tensor with $\epsilon_{12} = 1$ and 0 is the 2×2 null matrix.

The consistent quantization of the self-dual model is then obtained in terms of the following nonvanishing Dirac brackets:

$$\begin{aligned}\{f^0(x), f^i(y)\}_D &= -\frac{1}{m}\partial_i^x \delta^2(x-y) & \{f^i(x), f^j(y)\}_D &= -\epsilon^{ij}\delta^2(x-y) \\ \{f^i(x), A^j(y)\}_D &= -\epsilon^{ij}\delta^2(x-y) & \{A^i(x), \pi_j^A(y)\}_D &= \frac{1}{2}\delta_j^i \delta^2(x-y) \\ \{\pi_i^A(x), \pi_j^A(y)\}_D &= \frac{1}{4}\epsilon_{ij}\delta^2(x-y) & \{\pi_i^f(x), \pi_j^A(y)\}_D &= -\frac{1}{4}\epsilon_{ij}\delta^2(x-y) \\ \{f^i(x), \pi_j^f(y)\}_D &= \frac{1}{2}\delta_j^i \delta^2(x-y) & \{f^0(x), \pi_i^A(y)\}_D &= -\frac{1}{2m}\epsilon_{ij}\partial_x^j \delta^2(x-y) \\ \{A^0(x), \pi_0^A(y)\}_D &= \delta^2(x-y).\end{aligned}\quad (\text{A.5})$$

Now, let us extend phase space further to embed all the second-class constraints into the corresponding first-class ones, while in section 2 we partially embed them by eliminating the second-class ones originated from the symplectic structure of the Chern–Simons term.

To embed all the second-class constraints into the first-class ones by following the improved DQM as in section 2, we first introduce three pairs of auxiliary fields such as (θ^1, θ^2) , (σ^1, σ^2) and (ρ^1, ρ^2) satisfying the canonical Poisson brackets

$$\{\theta^1(x), \theta^2(y)\} = \{\sigma^1(x), \sigma^2(y)\} = \{\rho^1(x), \rho^2(y)\} = \delta^2(x-y) \quad (\text{A.6})$$

which define $\omega^{\alpha\beta}$ in (2.10) as

$$\omega^{\alpha\beta} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix}. \quad (\text{A.7})$$

From the strong involution relations $\{\tilde{\Omega}_\alpha, \tilde{\Omega}_\beta\} = 0$ and the ansatz of the form (2.13), we obtain a solution $X_{\alpha\beta}$ explicitly as

$$X_{\alpha\beta} = \begin{pmatrix} \sqrt{m} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{m} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (\text{A.8})$$

As a result, we convert all the second-class constraints to the first-class ones,

$$\tilde{\Omega}_i = \Omega_i + \sqrt{m}\theta^i \quad \tilde{\Omega}_{i+2} = \Omega_{i+2} + (-1)^i \sigma^i - \rho^i \quad \tilde{\Omega}_{i+4} = \Omega_{i+4} + \rho^i \quad (\text{A.9})$$

with $i, j = 1, 2$ satisfying the rank-0 algebra: $\{\tilde{\Omega}_\alpha, \tilde{\Omega}_\beta\} = 0$.

Similarly, we obtain for the improved first-class fields in the extended phase space

$$\begin{aligned} \tilde{f}^0 &= f^0 + \frac{1}{\sqrt{m}}\theta^2 & \tilde{f}^i &= f^i + \frac{1}{\sqrt{m}}\partial^i\theta^1 + (-1)^i\epsilon_{ij}\sigma^j \\ \tilde{A}^0 &= A^0 & \tilde{A}^i &= A^i + (-1)^i\epsilon_{ij}\sigma^j + \epsilon_{ij}\rho^j \\ \tilde{\pi}_0^f &= \pi_0^f + \sqrt{m}\theta^1 & \tilde{\pi}_i^f &= \pi_i^f + \frac{1}{2}(-1)^i\sigma^i - \frac{1}{2}\rho^i \\ \tilde{\pi}_0^A &= \pi_0^A & \tilde{\pi}_i^A &= \pi_i^A - \frac{1}{2\sqrt{m}}\epsilon_{ij}\partial^j\theta^1 + \frac{1}{2}\rho^i. \end{aligned} \quad (\text{A.10})$$

From these, it can readily be shown that in the master self-dual model the Poisson brackets in the extended phase space are exactly equivalent to the Dirac brackets (A.5) [8, 28].

On the other hand, since an arbitrary functional of the improved first-class fields is also first class, we can also directly obtain the desired first-class Hamiltonian⁴ $\tilde{\mathcal{H}}$ corresponding to the Hamiltonian \mathcal{H}_p in equation (2.5) via the substitution $f^\mu \rightarrow \tilde{f}^\mu$, $A^\mu \rightarrow \tilde{A}^\mu$, $\pi_\mu^f \rightarrow \tilde{\pi}_\mu^f$ and $\pi_\mu^A \rightarrow \tilde{\pi}_\mu^A$:

$$\begin{aligned} \tilde{\mathcal{H}}_p &= \mathcal{H}_c + \sqrt{m}\theta^1\partial_i f^i + \theta^2\left(-\sqrt{m}f^0 + \frac{1}{\sqrt{m}}\epsilon_{ij}\partial^i A^j\right) + mf^i(-1)^i\epsilon_{ij}\sigma^j \\ &\quad - (-1)^i f^0\partial_i\sigma^i + (f^0 - A^0)\partial_i\rho^i - \sqrt{m}\theta^1(-1)^i\epsilon_{ij}\partial^i\sigma^j \\ &\quad + \theta^2\left(-\frac{1}{2}\theta^2 + \frac{1}{\sqrt{m}}\partial_i\rho^i - (-1)^i\frac{1}{\sqrt{m}}\partial_i\sigma^i\right) + \frac{m}{2}(\sigma^i)^2 - \frac{1}{2}\partial_i\theta^1\partial^i\theta^1. \end{aligned} \quad (\text{A.11})$$

Next, we construct the Hamilton equations of motion for these first-class fields,

$$\begin{aligned} \frac{d}{dt}\tilde{f}^0 &= \partial^i\tilde{f}^i & \frac{d}{dt}\tilde{f}^i &= -m\epsilon^{ij}\tilde{f}^j + \frac{1}{m}\partial^i(\epsilon_{jk}\partial^j\tilde{A}^k) \\ \frac{d}{dt}\tilde{A}^0 &= 0 & \frac{d}{dt}\tilde{A}^i &= -m\epsilon^{ij}\tilde{f}^j + \partial^i\tilde{A}^0. \end{aligned} \quad (\text{A.12})$$

Here note that the equation of motion for \tilde{f}^0 in equation (A.12) can be rewritten in terms of covariant form, $\partial_\mu\tilde{f}^\mu = 0$, from which one can obtain the explicit form for \tilde{f}^μ , the duality relation [26]: $\tilde{f}^\mu = \frac{1}{m}\epsilon^{\mu\nu\rho}\partial_\nu\tilde{A}_\rho$, now written in terms of the improved first-class (gauge invariant) fields in the extended phase space.

Appendix B. BFT construction of involutive Hamiltonian

It is instructive to construct \tilde{H}_c obtained in (2.16), following the usual BFT procedure for obtaining the involutive Hamiltonian directly [6]. The procedure assumes that the involutive Hamiltonian can be written as the infinite series,

$$\tilde{H} = H_c + \sum_{n=1}^{\infty} H^{(n)} \quad H^{(n)} \sim (\Phi^i)^n \quad (\text{B.1})$$

⁴ Similar to section 2, we can also construct the involutive Hamiltonian \tilde{H} directly by following the method of Batalin *et al* [6] and thus demonstrate the equivalence of $\tilde{\mathcal{H}}_p$ up to total derivatives.

satisfying the initial condition $\tilde{H}(f^\mu, A^\mu, \pi_0^f, \pi_0^A; \Phi^i = 0) = H_c$. The general solution [6] for the involutive Hamiltonian \tilde{H} is then given by

$$H^{(n)} = -\frac{1}{n} \int d^2x d^2y \Phi^i(x) \epsilon_{ij} X^{jk}(x, y) G_k^{(n-1)}(y) \quad (\text{B.2})$$

where the generating functionals $G^{(n)}$ are

$$G_i^{(0)} = \{\Omega_i^{(0)}, H_c\} \quad G_i^{(n)} = \{\Omega_i^{(0)}, H^{(n)}\}_{\mathcal{O}} + \{\Omega_i^{(1)}, H^{(n-1)}\}_{\mathcal{O}}. \quad (\text{B.3})$$

Here the symbol \mathcal{O} denotes that the Poisson brackets are calculated among the original variables.

Explicit calculations for our model yield

$$G_1^{(0)} = m f^0 - \epsilon_{ij} \partial^i A^j \quad G_2^{(0)} = m \partial_i f^i \quad (\text{B.4})$$

which are substituted in (B.2) to obtain $H^{(1)}$:

$$H^{(1)} = \int d^2x \left[m \theta \partial_i f^i - \frac{1}{m} \pi_\theta (m f^0 - \epsilon_{ij} \partial^i A^j) \right]. \quad (\text{B.5})$$

The generating functionals for the next generation are

$$G_1^{(1)} = \pi_\theta \quad G_2^{(1)} = m \partial_i \partial^i \theta \quad (\text{B.6})$$

and yield

$$H^{(2)} = \int d^2x \left[-\frac{1}{2m} \pi_\theta^2 - \frac{m}{2} \partial_i \theta \partial^i \theta \right]. \quad (\text{B.7})$$

There are no further iterative higher order Hamiltonians, and thus the total Hamiltonian can be written as

$$\begin{aligned} \tilde{H} &= H_c + H^{(1)} + H^{(2)} \\ &= \int d^2x \left[-\frac{m}{2} f_\mu f^\mu + f^0 \epsilon_{ij} \partial^i A^j + A^0 \epsilon_{ij} (\partial^i f^j - \partial^j A^i) \right. \\ &\quad \left. + m \theta \partial_i f^i - f^0 \pi_\theta + \frac{1}{m} \pi_\theta \epsilon_{ij} \partial^i A^j - \frac{1}{2m} \pi_\theta^2 - \frac{m}{2} \partial_i \theta \partial^i \theta \right] \end{aligned} \quad (\text{B.8})$$

which is the same as the first-class Hamiltonian (2.16) up to a total derivative. This confirms the equivalence of the \tilde{H}_c (2.16) of the involutive Hamiltonian (B.8).

Appendix C. Hamiltonian–Dirac quantization of the symplectic Lagrangian

It is instructive to compare the gauge transformations (3.34), which are obtained from the analysis of the symplectic scheme, with the Hamiltonian description.

Appendix C.1. The original master Lagrangian

Our starting point is again the first-order Lagrangian (3.1). The canonical momenta conjugate to ξ_α are given by

$$\mathcal{P}_\alpha = a_\alpha \quad (\text{C.1})$$

and correspondingly we have six primary constraints, which we write in the canonical form

$$\phi_\alpha \equiv \mathcal{P}_\alpha - a_\alpha \approx 0. \quad (\text{C.2})$$

The corresponding primary Hamiltonian governing the time development of the system [1] is thus given by

$$\begin{aligned} H_p &= \sum_{\alpha} \int d^2x \mathcal{P}_{\alpha} \dot{\xi}_{\alpha}(x) - L + \sum_{\alpha} \int d^2x v_{\alpha} \phi_{\alpha} \\ &= V[\xi] + \sum_{\alpha} \int d^2x v_{\alpha} \phi_{\alpha} \end{aligned} \quad (\text{C.3})$$

where v_{α} are Lagrange multipliers (after suitable redefinition), and $V[\xi]$ is the potential (3.4).

With the above ‘canonical’ form for the *primary* constraints, we have for the corresponding Poisson brackets

$$\{\phi_{\alpha}(x), \phi_{\beta}(y)\} = \frac{\partial a_{\beta}(y)}{\partial \xi_{\alpha}(x)} - \frac{\partial a_{\alpha}(x)}{\partial \xi_{\beta}(y)} \equiv F_{\alpha\beta}^{(0)}(x, y). \quad (\text{C.4})$$

As we have seen, this matrix is not invertible, and possesses in fact ‘two’ zero modes⁵. They are obtained as usual by requiring the persistence in time of the primary constraints, and are found to be just $\varphi_1(z)$ and $\varphi_2(z)$, defined in equation (3.11). They represent the first generation of secondary constraints. There are no further (higher generation) constraints, and the algorithm ends at this point. We now collect all the constraints into a single ‘vector’,

$$(\Omega_A) = (\{\phi_{\alpha}\}, \varphi_1, \varphi_2). \quad (\text{C.5})$$

We correspondingly write for the second-class constraints

$$(\Omega_{\bar{A}}^{(2)}) = (\{\phi_{\alpha}\}, \varphi_1). \quad (\text{C.6})$$

The range of values that \bar{A} takes is implicit in the notation. It is readily recognized that, because of the ‘canonical’ form of the primary constraints,

$$\{\Omega_A(x), \Omega_B(y)\} = \begin{pmatrix} f(x, y) & \left(-\frac{\partial \varphi_2(y)}{\partial \xi_{\bar{A}}(x)}\right) \\ \left(\frac{\partial \varphi_2(y)}{\partial \xi_{\bar{B}}(x)}\right) & 0 \end{pmatrix} = F^{(1)}(x, y) \quad (\text{C.7})$$

with

$$f_{\bar{A}\bar{B}}(x, y) = \{\Omega_{\bar{A}}^{(2)}(x), \Omega_{\bar{B}}^{(2)}(y)\}$$

the elements of matrix (3.26). Since the submatrix f , being constructed from the second-class constraints, is invertible, we may define the ‘Dirac brackets’ in the conventional way, as

$$\begin{aligned} \{F, G\}_{\mathcal{D}} &= \{F, G\} - \sum_{\bar{A}, \bar{B}=1}^6 \int d^2z' \int d^2z \{F, \Omega_{\bar{A}}^{(2)}(z)\} f_{\bar{A}\bar{B}}^{-1}(z, z') \{\Omega_{\bar{B}}^{(2)}(z'), G\} \\ &= \int \int d^2z d^2z' \frac{\delta F}{\partial \xi_{\bar{A}}(z)} f_{\bar{A}\bar{B}}^{-1}(z, z') \frac{\delta G}{\partial \xi_{\bar{B}}(z')}. \end{aligned}$$

Hence the Dirac brackets coincide with the generalized Poisson brackets (3.31). In particular, choosing for F the coordinates $\xi_{\alpha_1}^{(1)}$, and for G the generator of the gauge transformation on the Hamiltonian level,

$$G = \int d^2y [\pi_0^A(y) \epsilon_1(y) + \varphi_2(y) \epsilon_2(y)]$$

we recover the gauge transformation (3.34) upon using the Lagrangian restriction on the gauge parameters discussed before.

⁵ In order to simplify the language, we do not say ‘two-fold infinity’ of zero modes.

Appendix C.2. The embedded master Lagrangian

From the Hamiltonian point of view, the symplectic Lagrangian (3.1) implies again the primary constraints (C.2) with (4.1)–(4.11), as well as two additional primary constraints

$$\phi^\theta \equiv \mathcal{P}_\theta - \pi_\theta \approx 0$$

and

$$\phi^{\pi_\theta} \equiv \mathcal{P}_{\pi_\theta} \approx 0$$

where $\mathcal{P}^\theta, \mathcal{P}^{\pi_\theta}$ are the momenta conjugate to θ and π_θ , respectively. The canonical Hamiltonian has the characteristic feature of being given just by the symplectic potential. Hence we have for the primary Hamiltonian

$$H_p = V[A, f, \pi_\theta, \theta] + \int d^2z \sum_\sigma v_\alpha \phi_\alpha.$$

The constraints $\phi_i^A \approx 0, \phi_i^f \approx 0, \phi^\theta \approx 0, \phi^\pi \approx 0$ fix the Lagrange multipliers $v_A^i = 0, v_f^i = 0, v_\theta = 0, v_\pi = 0$:

$$\begin{aligned} v_i^f &= -m\epsilon_{ij} f^j - \partial_i f^0 - \alpha\epsilon_{ij} \partial^j \theta & v_i^A &= -m\epsilon_{ij} f^j - \partial_i A^0 - \alpha\epsilon_{ij} \partial^j \theta \\ v^\theta &= \frac{1}{\beta}(\pi - \alpha f^0) & v^\pi &= -\alpha\partial_i f^i - \beta\partial_i \partial^i \theta. \end{aligned} \quad (\text{C.8})$$

We recognize that the elements of $F^{(0)}$ are just the Poisson brackets of the primary constraints: $F_{\alpha\beta}^{(0)} = \{\phi_\alpha, \phi_\beta\}$. The usual Dirac algorithm leads to the secondary constraints $\phi_a \approx 0, a = 1, 2$, which for $\alpha^2/\beta = m$ are both found to have identically vanishing Poisson brackets with the primary Hamiltonian, after making use of the explicit expressions for the fixed Lagrange parameters. Hence no new constraints are generated, and we have in the final stage two first-class primary and two first-class secondary constraints, generating in the usual manner the extended gauge symmetry of the Hamiltonian. It is interesting that in the symplectic approach we directly obtain the more restricted symmetry of the Lagrangian.

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